UDC 521.1

PERIODIC MOTIONS OF A RIGID BODY WITH A FIXED POINT IN THE GRAVITY FIELD OF TWO CENTERS

PMM Vol. 41, № 3, 1977, pp. 558-561
Iu. V. BARKIN and V. E. IEVLEV
(Moscow)
(Received September 14, 1976)

Andoyer variables [1] are used to investigate the problem of existence of periodic motions of a rigid body with a fixed point in the gravity field of two stationary centers.

1. Consider the motion of a rigid body about a fixed point O_1 in the Newtonian gravity field of two fixed centers M_1 and M_2. We introduce the following coordinate systems: $OXYZ$ is a stationary coordinate system chosen so that the centers of attraction $M_1(X_1, 0, 0)$ and $M_2(X_2, 0, 0)$ lie on the OX-axis and the fixed point $O_1(0, Y_0, 0)$ of the body M lies on the OY-axis; O_1xyz is a coordinate system with the origin at the fixed point O_1, the axes of which are parallel to the axes of the $OXYZ$-coordinate system; $O_1\xi\eta\zeta$ is a moving coordinate system the axes of which are directed along the principal axes of inertia of the body M with respect to the fixed point. We described the motion of the body using the Andoyer elements

$$L, G, H, l, g, h \tag{1.1}$$

Here G denotes the magnitude of the vector $\mathbf{G}$ of the moment of momentum of the body rotation; L and H are the projections of the vector $\mathbf{G}$ on the $O_1\zeta$-axis and O_1z-axis of the body, respectively; l is the angle counted from the line of intersection of the intermediate plane P normal to the vector $\mathbf{G}$ with the $O_1\xi\eta$ plane of the body, to the $O_1\xi$-axis; h is the angle counted from the O_1x-axis to the line of intersection of the O_1xy-plane and the plane P, g is the angle between the line of intersection of the plane O_1xy and the plane P, g is the angle between the line of intersection of the planes P, $O_1\xi\eta$ and the line of intersection of the planes P, O_1xy.

The motion in question is determined by the Hamiltonian

$$K = T - U_1 - U_2 \tag{1.2}$$

$$T = \frac{G^2 - L^2}{2AB}(A\cos^2 l + B\sin^2 l) + \frac{L^2}{2C}$$

$$U_s = -P_s(\xi_c\alpha_s + \eta_c\beta_s + \zeta_c\gamma_s) - \frac{3P_s}{2mR_s}(A\alpha_s^2 + B\beta_s^2 + C\gamma_s^2)$$

$$P_s = f\frac{m_s m}{R_s^2}, \quad s = 1, 2$$

Here f is the gravitational constant, m is the mass of the body, m_1 and m_2 are the masses of the fixed centers of attraction; A, B and C are the prinicpal moments of inertia of the body for the fixed point O_1; R_1 and R_2 are the corresponding distances from the centers M_1 and M_2 to the fixed point O_1; α_s, β_s and γ_s are the direction cosines of the line M_sO_1 in the moving coordinate system $O_1\xi\eta\zeta$; ξ_c, η_c and ζ_c are the coordinates of the center of mass of the body in the $O_1\xi\eta\zeta$ -coordinate system.

Let us find the force function of the problem, depending explicitly on the Andoyer elements (1.1). Using the formulas for the unperturbed motion [2], we express the direction cosines $\alpha_s, \beta_s, \gamma_s$ in terms of the Andoyer elements. After the transformations we obtain the force function in the form of a trigonometric series. We assume that the body is almost axisymmetric and its fixed point lies near its center of mass. In this case the Hamiltonian (1.2) can be written in the form which allows the application of the Poincaré method of small parameter [3]

$$K = K_0 + \mu K_1, \quad K_0 = \frac{G^2 - L^2}{2A} + \frac{L^2}{2C} \tag{1.3}$$

$$\mu K_1 = \frac{A-B}{2AB}(G^2 - L^2)\cos^2 l - U_1 - U_2$$

Here K_0 is the Hamiltonian defining the generating solution and μK_1 is the perturbing Hamiltonian.

The following quantity serves as the small parameter

$$\mu = \max\left\{\frac{|A-B|}{A}, \frac{A}{mY_0^2}, \frac{B}{mY_0^2}, \frac{|C}{mY_0^2}, \frac{\xi_c}{Y_0}, \frac{\eta_c}{Y_0}, \frac{\zeta_c}{Y_0}\right\}$$

Here Y_0 is the coordinate of the point O_1 in the $OXYZ$ -coordinate system.
Let us write the simplified system of equations of motion

$$L^{\cdot} = 0, \quad G^{\cdot} = 0, \quad H^{\cdot} = 0$$

$$l^{\cdot} = \frac{A-C}{AC}L, \quad g^{\cdot} = \frac{G}{A}, \quad h^{\cdot} = 0$$

Its general solution has the form

$$L = L_0 = \text{const}, \quad G = G_0 = \text{const}, \quad H = H_0 = \text{const} \tag{1.4}$$

$$l = n_1 t + l_0, \quad g = n_2 t + g_0, \quad h = h_0$$

$$n_1 = \frac{A-C}{AC}L_0; \quad n_2 = \frac{G_0}{A}$$

The solution (1.4) will be periodic if for any integers k_1 and k_2 we have $k_1 n_1 = k_2 n_2$.

The period of the generating solution is

$$T = 2\pi k_1 / n_2 = 2\pi k_2 / n_2 \tag{1.5}$$

2. We shall prove the existence of periodic solutions with the period (1.5) of the system of equations with the Hamiltonian (1.3), coinciding with the generating solution at $\mu = 0$.

In accordance with the Poincaré's theory of periodic solutions, the equations of motion with the Hamiltonian (1.3) admit T-periodic solutions for small values of the parameter μ, provided that the corresponding generating solutions satisfy the following group of conditions:

$$\Delta_1(K_0) \neq 0 \tag{2.1}$$
$$\frac{\partial [K_1]}{\partial g_0} = \frac{\partial [K_1]}{\partial h_0} = 0, \quad \frac{\partial [K_1]}{\partial H_0} = 0$$
$$\Delta_2([K_1]) \neq 0, \quad [K_1] = \frac{1}{T}\int_0^T K_1 dt$$

Here Δ_1 is the Hesse determinant of K_0 relative to G_0 and L_0, and Δ_2 is the Hesse determinant of $[K_1]$ relative to l_0, h_0 and H_0. The first condition holds for any generating solution when $A \neq C$, since

$$\Delta_1(K_0) = \frac{A-C}{A^2C}$$

To assess the remaining conditions, we must calculate the value of $[K_1]$. The following cases are possible:

$$|k_1| = |k_2|; \quad |k_1| = 1, \quad |k_2| = 2$$
$$|k_1| = 2, \quad |k_2| = 1; \quad |k_1| + |k_2| \geqslant 4$$

For $|k_1| + |k_2| \geqslant 4,\ |k_1| \geqslant 2$ the last condition of (2.1) fails since $[K_1]$ is independent of h_0 and g_0. In this case the search for the existence of periodic solutions necessitates the inclusion of terms of higher order of smallness in the expansion of the force function. We shall limit ourselves to the case of commensurability with $k_1 = k_2$. We assume that the centers M_1 and M_2 are of equal mass and are situated at the complex conjugate distances

$$m_1 = m_2 = m_0, \quad X_{1,2} = \pm id, \quad d = \text{const}, \quad i = \sqrt{-1}$$

As we know [4, 5], the force function can in this case approximate, with a high degree of accuracy, the potential of a spheroid. We shall also assume that the center of mass of the body M coincides with the fixed point O_1.

Under these assumptions we obtain

$$\mu[K_1] = \frac{A-B}{4AB}(G_0^2 - L_0^2) + (\varkappa_1 + Y_0^2\varkappa_2) C_{000}^1 + \tag{2.2}$$
$$(\varkappa_1 + Y_0^2\varkappa_2) C_{2,-2,0}^1 \cos 2g_0 + (\varkappa_1 - Y_0^2\varkappa_2)\cos 2h_0 +$$
$$(\varkappa_1 - Y_0^2\varkappa_2) C_{2,-2,2}^1 \cos(2g_0 - 2h_0) + (\varkappa_1 - Y_0^2\varkappa_2) C_{2,-2,-2}^1 \cos(2g_0 + 2h_0)$$

where

$$C_{000}^1 = (B - A)\left[{}^1/_8(1 + \cos^2\theta)(1 + \cos^2\rho)\right] +$$
$$(C - A)\left[{}^1/_4 \sin^2\theta\,(1 + \cos^2\rho) + {}^1/_2 \sin^2\rho\cos^2\theta\right]$$

$$C^1_{002} = {}^1/_8\,(A - B)\sin^2\rho\,(\sin^2\theta - 2\cos^2\theta) + {}^1/_4\,(A - C)\,(2 - 3\sin^2\theta)$$
$$C^1_{2,-2,0} = {}^1/_{16}\,(A - B)\sin^2\rho\,(1 - \cos\theta)^2$$
$$C^1_{2,-2,\pm 2} = {}^1/_8\,(A - B)\sin^2\theta\,/\,2\,(1 \pm \cos\rho)^2$$
$$\varkappa_1 = f\,\frac{3d^2 m_0}{R^5}, \quad \varkappa_2 = -\frac{\varkappa_1}{d^2}, \quad R = \sqrt{Y_0^2 - d^2}$$
$$\cos\theta = G\,/\,L, \quad \cos\rho = G\,/\,H$$

The second condition of (2.1) can now be easily written in the explicit form. As the result, we find the generating values of the angular variables

$$l_0 = 0; \quad g_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}; \quad h_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \tag{2.3}$$

Using the above values, we transform the third equation of (2.1) to the form

$$f_1(\theta)\cos\rho = 0 \tag{2.4}$$
$$f_1(\theta) = M_2\sin^4\theta\,/\,2 + M_1\sin^2\theta\,/\,2 + M_0$$

where M_0, M_1 and M_2 are known constants. Equation (2.4) is certainly satisfied when $\rho = \pi\,/\,2$. The solution $\rho = \pi\,/\,2$ together with the solution $h_0 = 0, \pi\,/\,2, \pi, 3\pi\,/\,2$ admits a simple geometrical interpretation: the vector $\mathbf{G}$ is either parallel, or perpendicular to the segment M_1M_2. The last condition of (2.1) becomes

$$f_1(\theta)\,f_2(\theta)\sin^4\frac{\theta}{2} \neq 0, \quad f_2(\theta) = N_2\sin^4\frac{\theta}{2} + N_1\sin^2\frac{\theta}{2} + N_0 \tag{2.5}$$

The constants N_0, N_1 and N_2 are also known. The condition (2.5) fails at a finite number of points. It fails at $\theta = 0$, and at the points given by the equation

$$f_1(\theta) f_2(\theta) = 0$$

Thus we have shown that in the case in question the problem of motion of a rigid body about a fixed point in the gravity field of two fixed centers admits a family of periodic Poicaré solutions. In the generating solution we can choose arbitrarily the magnitude G of the vector $\mathbf{G}$, the projection L of the vector $\mathbf{G}$ on the l_0 axis of the body, the value of the elements $O_1\zeta$ and the initial instant of time.

REFERENCES

1. Andoyer, M. H., Cours de mécanique céleste. Paris, Vol. 1, 1923.
2. Barkin, Iu. V., Equations of perturbed rotation of a rigid body about its center of mass. Vestn. MGU, Ser. fiz., astron., № 1, 1975.

3. Poincaré, A., Collected Works, Vol. 1, Moscow, "Nauka", 1971.
4. Aksenov, E. P., Grebenikov, E. A. and Demin, V. G., General solution of the problem of motion of an artificial satellite in the normal, Earth's gravity field. In coll: Artificial Earth's Satellites. Ed. 8, M., Izd. Akad. Nauk SSSR, 1961.
5. Demin, V. G., Motion of an Artificial Satellite in a Noncentral Gravity Field. Moscow, "Nauka", 1968.

Translated by L. K.